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The group classification of a scalar stochastic differential equation

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Abstract
Lie point group classification of a scalar stochastic differential equation (SDE) with one-dimensional Brownian motion is presented. The admitted symmetry group can be zero, one, two or three dimensional. If an equation admits a three-dimensional symmetry group, it can be transformed into the equation of Brownian motion by a change of variables with non-random time transformation. Such equations can be integrated by quadratures. There are described drift coefficients for which SDEs with constant diffusion coefficients admit two- and three-dimensional symmetry groups.

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1. Introduction

The Lie group theory of differential equations has been well understood [1–3]. It studies transformations by taking solutions of differential equations into other solutions of the same equations. This theory is a very general and useful tool for finding analytical solutions of large classes of differential equations.

The application of the Lie group theory to stochastic differential equations (SDEs) is much more recent. First, restricted cases of point transformations were considered [4–7]. Then, the theory for general point transformations was developed [8–12]. In the latter case, the transformation of the Brownian motion needs to be more deeply specified.

In this paper we present the Lie group classification of a scalar stochastic ODE with one-dimensional Brownian motion. The admitted symmetry group can be zero, one, two or three dimensional. If an equation admits a three-dimensional symmetry group, it can be transformed into the equation of Brownian motion by a point change of variables with non-random time transformation. Therefore, for such equations one can write down closed-form solutions using quadratures. The equations with two- and three-dimensional symmetry groups are considered in greater detail. Taking a constant diffusion coefficient, we characterize the drift coefficients for which SDEs admit two- and three-dimensional symmetry groups. Transformations of these invariant equations into the corresponding canonical forms are provided.

It should be noted that the paper deals with infinitesimal Lie group transformations which preserve the form of SDEs. Reconstruction of finite transformations from infinitesimal ones was discussed in [10, 12]. Generally, it is not guaranteed that the finite transformations, which are recovered from infinitesimal transformations, transform solutions of SDEs into another solutions.

2. Scalar SDEs and symmetries

Let us consider the Itô stochastic differential equation

$$\mathrm{d}x = f(t,x)\,\mathrm{d}t + g(t,x)\,\mathrm{d}W(t), \qquad g(t,x) \not\equiv 0, \tag{2.1}$$

where $f(t, x)$ is a drift, $g(t, x)$ is a diffusion and $W(t)$ is a standard Wiener process [13, 14]. We will be interested in infinitesimal group transformations (near identity change of variables)

$$\bar{t} = \bar{t}(t,x,a) \approx t + \tau(t,x)a, \qquad \bar{x} = \bar{x}(t,x,a) \approx x + \xi(t,x)a, \tag{2.2}$$

which leave equation (2.1) and the framework of Itô calculus invariant. Transformations (2.2) can be represented by generating operators of the form

$$X = \tau(t,x)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}. \tag{2.3}$$

The determining equations for admitted symmetries are (see, for example, [9])

$$\xi_t + f\xi_x - \xi f_x - \tau f_t - f\tau_t - f^2\tau_x - \tfrac{1}{2}fg^2\tau_{xx} + \tfrac{1}{2}g^2\xi_{xx} = 0, \tag{2.4}$$

$$g\xi_x - \xi g_x - \tau g_t - \frac{g}{2}\left(\tau_t + f\tau_x + \frac{1}{2}g^2\tau_{xx}\right) = 0, \tag{2.5}$$

$$g\tau_x = 0. \tag{2.6}$$

It is interesting to note that the determining equations are deterministic even though they describe symmetries of a stochastic differential equation.

In the general case, when functions $f(t, x)$ and $g(t, x)$ are arbitrary, the determining equations (2.4)–(2.6) have no non-trivial solutions, i.e. there are no symmetries.

The last determining equation (2.6) can be solved as

$$\tau = \tau(t). \tag{2.7}$$

Therefore, the symmetries admitted by equation (2.1) are fiber-preserving symmetries

$$X = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x} \tag{2.8}$$

that substantially simplifies further consideration. In particular, we are restricted to the equivalence transformations

$$\bar{t} = \bar{t}(t), \qquad \bar{x} = \bar{x}(t,x), \qquad \bar{t}_t \neq 0, \qquad \bar{x}_x \neq 0, \tag{2.9}$$

where change of time is not random. According to the general result concerning with the random time change in Brownian motion [14], the Brownian motion is transformed as

$$\mathrm{d}\bar{W}(\bar{t}) = \sqrt{\frac{\mathrm{d}\bar{t}(t)}{\mathrm{d}t}}\,\mathrm{d}W(t). \tag{2.10}$$

Remark 2.1. Because of (2.7) the symmetries admitted by equation (2.1) form a Lie algebra. It was noted in [9] (see also [8]) that in general symmetries of Itô systems do not form Lie algebras while symmetries of Stratonovich systems always do. In a particular case $\tau = \tau(t)$ the determining equations for corresponding Itô and Stratonovich systems are identical. Therefore, all results of this paper established for Itô SDE (2.1) are also valid for the corresponding Stratonovich SDE:

$$\mathrm{d}x = h(t, x)\,\mathrm{d}t + g(t, x) \circ \mathrm{d}W(t), \qquad h = f - \tfrac{1}{2} g g_x. \tag{2.11}$$

Let us illustrate symmetry properties by an example.

Example 2.1. The equation

$$\mathrm{d}x = f(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t) \qquad g(t) > 0 \tag{2.12}$$

can be solved as

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s)\,\mathrm{d}s + \int_{t_0}^{t} g(s)\,\mathrm{d}W(s). \tag{2.13}$$

Scalar stochastic differential equations which can be transformed by a non-degenerate change of variables into this equation can be easily integrated by quadratures, i.e. they have closed-form solutions. A wider family of stochastic differential equations can be integrated if we consider representations of SDEs by decoupled systems [15, 16]. Since this integration method goes beyond transformations (2.9), it will not be considered in this paper. It is worth mentioning that besides quadratures there are other means to present closed-form solutions of stochastic differential equations [17].

Equation (2.12) admits symmetries (cf [12])

$$X_1 = \frac{1}{g^2}\frac{\partial}{\partial t} + \frac{f}{g^2}\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial x},$$
$$X_3 = \left(\frac{2}{g^2}\int g^2\,\mathrm{d}t\right)\frac{\partial}{\partial t} + \left(x + \frac{2f}{g^2}\int g^2\,\mathrm{d}t - \int f\,\mathrm{d}t\right)\frac{\partial}{\partial x}$$

with the Lie algebra structure

$$[X_1, X_2] = 0, \qquad [X_1, X_3] = 2X_1, \qquad [X_2, X_3] = X_2.$$

Under the change of variables

$$\bar{t} = \int g^2(t)\,\mathrm{d}t, \qquad \bar{x} = x - \int f(t)\,\mathrm{d}t \tag{2.14}$$

equation (2.12) is transformed into the equation of Brownian motion:

$$\mathrm{d}\bar{x} = \mathrm{d}\bar{W}(\bar{t}), \tag{2.15}$$

which admits the symmetries

$$\bar{X}_1 = \frac{\partial}{\partial \bar{t}}, \qquad \bar{X}_2 = \frac{\partial}{\partial \bar{x}}, \qquad \bar{X}_3 = 2\bar{t}\frac{\partial}{\partial \bar{t}} + \bar{x}\frac{\partial}{\partial \bar{x}}.$$

Solution (2.13) of the original equation can be obtained from the solution

$$\bar{x}(\bar{t}) = \bar{x}(\bar{t}_0) + \bar{W}(\bar{t}) - \bar{W}(\bar{t}_0) \tag{2.16}$$

of the Brownian motion equation by change of variables (2.14).

3. Group classification

Similarly to the treatment of the one-dimensional Fokker–Planck (FP) equation, corresponding to SDE (2.1), in [18] it is useful to prove the following result.

Theorem 3.1. *If equation (2.1) admits a symmetry operator (2.8), then there exists a change of variables*

$$\bar{t} = \bar{t}(t), \qquad \bar{x} = \bar{x}(t, x)$$

which transforms the equation into the form

$$\mathrm{d}\bar{x} = \bar{f}(\bar{x})\,\mathrm{d}\bar{t} + \bar{g}(\bar{x})\,\mathrm{d}\bar{W}(\bar{t}), \qquad g(\bar{x}) \not\equiv 0. \tag{3.1}$$

Proof. There will be considered two cases: $\tau(t) \not\equiv 0$ and $\tau(t) \equiv 0$.

(1) $\tau(t) \not\equiv 0$.

Let $\tau(t) \not\equiv 0$ and $\xi(t, x)$ be solutions of the determining equations (2.4)–(2.6). We consider the change of variables

$$\bar{t} = \bar{t}(t) = \int \frac{\mathrm{d}t}{\tau(t)}, \qquad \bar{x} = \bar{x}(t, x), \qquad \bar{x}_x \neq 0,$$

where $\bar{x}(t, x)$ is a solution of the differential equation:

$$\tau(t)\bar{x}_t + \xi(t, x)\bar{x}_x = 0. \tag{3.2}$$

In the new variables $\bar{t}$ and $\bar{x}$ the symmetry operator takes the form

$$\bar{X} = \frac{\partial}{\partial \bar{t}}.$$

By the rules of Itô calculus

$$\mathrm{d}\bar{x} = \left(\bar{x}_t + \bar{x}_x f + \tfrac{1}{2}\bar{x}_{xx} g^2\right)\mathrm{d}t + \bar{x}_x g\,\mathrm{d}W(t).$$

Changing differentials (see (2.10)), we obtain the transformed equation

$$\mathrm{d}\bar{x} = \bar{f}(\bar{t}, \bar{x})\,\mathrm{d}\bar{t} + \bar{g}(\bar{t}, \bar{x})\,\mathrm{d}\bar{W}(\bar{t}) \tag{3.3}$$

with the drift and diffusion coefficients

$$\bar{f}(\bar{t}, \bar{x}) = \left(\bar{x}_t + \bar{x}_x f + \tfrac{1}{2}\bar{x}_{xx} g^2\right)\tau, \qquad \bar{g}(\bar{t}, \bar{x}) = \bar{x}_x g\sqrt{\tau}, \tag{3.4}$$

which should be expressed in terms of the new variables $\bar{t}$ and $\bar{x}$. We assume that $\tau > 0$ (one can always choose the symmetry so that $\tau(t) > 0$ on some interval where $\tau \neq 0$). Let us show that the transformed equation (3.3) with coefficients (3.4) has the form (3.1)

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}}\bar{f}(\bar{t}, \bar{x}) &= \left(\tau(t)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x}\right)\left[\left(\bar{x}_t + \bar{x}_x f + \frac{1}{2}\bar{x}_{xx} g^2\right)\tau\right] \\
&= \tau\,(\tau\bar{x}_t + \xi\bar{x}_x)_t + f\tau\,(\tau\bar{x}_t + \xi\bar{x}_x)_x + \frac{g^2}{2}\tau\,(\tau\bar{x}_t + \xi\bar{x}_x)_{xx} \\
&\quad - \bar{x}_x\tau\left(\xi_t + f\xi_x - \xi f_x - \tau f_t - f\tau_t + \frac{1}{2}g^2\xi_{xx}\right) \\
&\quad - \bar{x}_{xx}\tau g\left(g\xi_x - \xi g_x - \tau g_t - \frac{g}{2}\tau_t\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}}\bar{g}(\bar{t}, \bar{x}) &= \left(\tau(t)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x}\right)[\bar{x}_x g\sqrt{\tau}] \\
&= g\sqrt{\tau}(\tau\bar{x}_t + \xi\bar{x}_x)_x - \bar{x}_x\sqrt{\tau}\left(g\xi_x - \xi g_x - \tau g_t - \frac{g}{2}\tau_t\right) = 0.
\end{aligned}$$

For the last equalities we used equation (3.2) and determining equations (2.4), (2.5).

(2) $\tau(t) \equiv 0$.

If $\tau(t) \equiv 0$ and $\xi(t, x) \not\equiv 0$, we can choose change of variables

$$\bar{t} = t, \qquad \bar{x} = \int \frac{\mathrm{d}x}{\xi(t, x)}$$

to bring the equation (2.1) to the form (3.3) with

$$\bar{f}(\bar{t}, \bar{x}) = -\int \frac{\xi_t}{\xi^2}\,\mathrm{d}x + \frac{f}{\xi} - \frac{g^2}{2}\frac{\xi_x}{\xi^2}, \qquad \bar{g}(\bar{t}, \bar{x}) = \frac{g}{\xi}, \tag{3.5}$$

where the drift and diffusion should be expressed in terms of the new variables. Using

$$\begin{aligned}\frac{\partial}{\partial \bar{x}}\bar{f}(\bar{t}, \bar{x}) &= \left(\xi(t, x)\frac{\partial}{\partial x}\right)\left[-\int \frac{\xi_t}{\xi^2}\,\mathrm{d}x + \frac{f}{\xi} - \frac{g^2}{2}\frac{\xi_x}{\xi^2}\right] \\ &= -\frac{1}{\xi}\left(\xi_t + f\xi_x - \xi f_x + \frac{1}{2}g^2\xi_{xx}\right) - \frac{g\xi_x}{\xi^2}(\xi g_x - g\xi_x) = 0\end{aligned}$$

and

$$\frac{\partial}{\partial \bar{x}}\bar{g}(\bar{t}, \bar{x}) = \left(\xi(t, x)\frac{\partial}{\partial x}\right)\left[\frac{g}{\xi}\right] = \frac{1}{\xi}(\xi g_x - g\xi_x) = 0,$$

where the last identities follow from the determining equations for the coefficients of the symmetry, we conclude that the transformed equation has the form

$$\mathrm{d}\bar{x} = \bar{f}(\bar{t})\,\mathrm{d}\bar{t} + \bar{g}(\bar{t})\,\mathrm{d}\bar{W}(\bar{t}). \tag{3.6}$$

This equation was considered in example 2.1, where we showed that it can be transformed into the equation of Brownian motion, i.e. an equation with time-independent coefficients. □

Theorem 3.1 allows us to restrict further consideration to the stochastic differential equation

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}W(t), \qquad g(x) \not\equiv 0. \tag{3.7}$$

By change of variables

$$x \rightarrow \int \frac{\mathrm{d}x}{g(x)},$$

this equation can be simplified further. We obtain an equation of the form

$$\mathrm{d}x = f(x)\,\mathrm{d}t + \mathrm{d}W(t), \tag{3.8}$$

for which group classification can be easily performed. From the determining equations (2.4)–(2.6) we get

$$\tau(t), \qquad \xi(t, x) = \frac{\tau'(t)}{2}x + A(t), \tag{3.9}$$

$$\left(\frac{\tau'(t)}{2}x + A(t)\right)f'(x) + \frac{\tau'(t)}{2}f(x) = \frac{\tau''(t)}{2}x + A'(t), \tag{3.10}$$

where $A(t)$ is an arbitrary function. Differentiating (3.10) twice with respect to x, we obtain

$$\frac{\tau'(t)}{2}(xf'''(x) + 3f''(x)) + A(t)f'''(x) = 0. \tag{3.11}$$

Symmetries different from time translation ($\tau'(t) \not\equiv 0$ or $A(t) \not\equiv 0$) can exist if

$$f'''(x) = 0 \qquad \text{or} \qquad f''(x) = \frac{C_1}{(x + C_2)^3},$$

where C_1 and C_2 are arbitrary constants. Direct consideration of all possible subcases, which can lead to non-trivial solutions for $\tau(t)$ and $A(t)$, provides us with the following Lie group classification.

(1) Arbitrary $f(x)$

$$X_1 = \frac{\partial}{\partial t}.$$

(2) $f(x) = \dfrac{\alpha}{x + \gamma} + \beta(x + \gamma), \alpha \neq 0$

(a) $\beta \neq 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \mathrm{e}^{2\beta t}\frac{\partial}{\partial t} + \beta(x + \gamma)\,\mathrm{e}^{2\beta t}\frac{\partial}{\partial x};$$

(b) $\beta = 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 2t\frac{\partial}{\partial t} + (x + \gamma)\frac{\partial}{\partial x}.$$

(3) $f(x) = \alpha x + \beta$

(a) $\alpha \neq 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \mathrm{e}^{2\alpha t}\frac{\partial}{\partial t} + (\alpha x + \beta)\,\mathrm{e}^{2\alpha t}\frac{\partial}{\partial x}, \qquad X_3 = \mathrm{e}^{\alpha t}\frac{\partial}{\partial x};$$

(b) $\alpha = 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 2t\frac{\partial}{\partial t} + (x + \beta t)\frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial x}.$$

Here α, β and γ are constants. The equations of subcases 2a and 2b can be transformed to the equation

$$\mathrm{d}\bar{x} = \frac{\alpha}{\bar{x}}\,\mathrm{d}\bar{t} + \mathrm{d}\bar{W}(\bar{t})$$

by change of variables

$$\bar{t} = t_0 - \frac{\mathrm{e}^{-2\beta t}}{2\beta}, \qquad \bar{x} = \mathrm{e}^{-\beta t}(x + \gamma)$$

and

$$\bar{t} = t, \qquad \bar{x} = x + \gamma,$$

respectively. Therefore, they represent the same equivalence class. Similarly, the equations of subcases 3a and 3b can be transformed to the equation of Brownian motion (2.15) by change of variables

$$\bar{t} = t_0 - \frac{\mathrm{e}^{-2\alpha t}}{2\alpha}, \qquad \bar{x} = \mathrm{e}^{-\alpha t}\left(x + \frac{\beta}{\alpha}\right)$$

and

$$\bar{t} = t, \qquad \bar{x} = x - \beta t.$$

Finally, the group classification of scalar SDE (2.1) with respect to equivalence transformations (2.9) is presented in table 1.

The results can be summed up as a theorem.

Table 1. Lie group classification of a scalar stochastic differential equation.

Group dimension	Basis operators	Equation
0	No symmetries	$\mathrm{d}x = f(t,x)\,\mathrm{d}t + \mathrm{d}W(t)$
1	$X_1 = \dfrac{\partial}{\partial t}$	$\mathrm{d}x = f(x)\,\mathrm{d}t + \mathrm{d}W(t)$
2	$X_1 = \dfrac{\partial}{\partial t}, X_2 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x}$	$\mathrm{d}x = \dfrac{\alpha}{x}\mathrm{d}t + \mathrm{d}W(t), \alpha \neq 0$
3	$X_1 = \dfrac{\partial}{\partial t}, X_2 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x}, X_3 = \dfrac{\partial}{\partial x}$	$\mathrm{d}x = \mathrm{d}W(t)$

Theorem 3.2. *The dimension of the Lie algebra of symmetries admitted by equation (2.1) can be equal to 0, 1, 2 or 3. If the equation admits a three-dimensional symmetry group, it can be transformed into the equation of Brownian motion by change of variables (2.9).*

Comparing the results of the group classification and the discussion concerning integrability by quadratures (example 2.1), we can formulate a sufficient condition of integrability by quadratures as follows.

Corollary 3.3. *The stochastic differential equation (2.1) is integrable by quadratures if it admits a three-dimensional symmetry group.*

The structure of the Lie algebras present in the group classification allows us to state the following result.

Theorem 3.4. *The stochastic differential equation (2.1) admits a three-dimensional symmetry group if and only if it admits a symmetry of the form*

$$X_* = \xi(t,x)\frac{\partial}{\partial x}. \tag{3.12}$$

A SDE admits such symmetry if and only if functions f(t, x) and g(t, x) satisfy the condition

$$\left(\frac{g_t}{g} - g\left(\frac{f}{g}\right)_x + \frac{1}{2}gg_{xx}\right)_x = 0. \tag{3.13}$$

Proof. Taking into account how symmetries of the representative equations, given in table 1, are transformed by changes of variables (2.9), we see that the equivalence class of the Brownian motion equation can be characterized as SDEs admitting symmetry (3.12). Condition (3.13) is obtained from the determining equations. □

For Stratonovich equation (2.11) condition (3.13) gets simplified as

$$\left(\frac{g_t}{g} - g\left(\frac{h}{g}\right)_x\right)_x = 0. \tag{3.14}$$

4. Criterion of invariance under two- and three-dimensional symmetry groups

In the previous section we provided Lie group classification of a scalar SDE. To obtain these results existence of one admitted symmetry was assumed. This symmetry was used in the simplifying transformation, which lead us to equation (3.7) used further for the simplification and classification. It can be useful to have a criterion which does not assume any knowledge concerning the admitted symmetry group. Such criterion for invariance under two- and three-dimensional symmetry groups will be established in this section.

It was stated in theorem 3.4 that if the stochastic equation (2.1) admits a symmetry (3.12), then it admits a three-dimensional symmetry group. This observation can be used to establish a criterion when the admitted symmetry group is two dimensional. Such groups consist of two symmetry operators X_1 and X_2 which satisfy the condition $\tau(t) \not\equiv 0$.

Let us establish when the stochastic differential equation

$$\mathrm{d}x = f(t,x)\,\mathrm{d}t + \mathrm{d}W(t) \tag{4.1}$$

admits two- and three-dimensional symmetry groups. This consideration is similar to the treatment of one-dimensional Fokker–Planck equation in [18]. It should be noted that any diffusion process can be reduced to a process governed by equation (4.1) with the help of a random replacement of time [19].

The determining equations (2.4)–(2.6) for equation (4.1) lead to

$$\tau(t), \qquad \xi(t,x) = \frac{\tau'(t)}{2}x + A(t), \tag{4.2}$$

$$\tau(t) f_t + \left(\frac{\tau'(t)}{2}x + A(t)\right) f_x + \frac{\tau'(t)}{2} f = \frac{\tau''(t)}{2}x + A'(t), \tag{4.3}$$

where $A(t)$ is an arbitrary function. It is clearly seen that the admitted group can be at most three dimensional.

Let us find functions $f(t,x)$ for which there exist two linearly independent solutions $\tau_1(t)$ and $\tau_2(t)$. Differentiating (4.3) twice with respect to x, we get

$$\tau(t) f_{txx} + \left(\frac{\tau'(t)}{2}x + A(t)\right) f_{xxx} + \frac{3}{2}\tau'(t) f_{xx} = 0. \tag{4.4}$$

If we assume that $f_{xxx} = 0$, then $f_{xx} = h(t)$ and equation (4.4) takes the form

$$\tau(t)h'(t) + \tfrac{3}{2}\tau'(t)h(t) = 0. \tag{4.5}$$

If $h(t) \not\equiv 0$, there is only one linearly independent solution $\tau(t)$. For $h(t) \equiv 0$ equation (4.5) is identically satisfied so that the case $f_{xx} = 0$ requires a separate treatment.

We proceed with $f_{xxx} \neq 0$. Rearranging (4.4), we get

$$A(t) = -\tau(t)\frac{f_{txx}}{f_{xxx}} - \frac{\tau'(t)}{2}\frac{x f_{xxx} + 3 f_{xx}}{f_{xxx}},$$

which leads to

$$\tau(t)\left(\frac{f_{txx}}{f_{xxx}}\right)_x + \frac{\tau'(t)}{2}\left(\frac{x f_{xxx} + 3 f_{xx}}{f_{xxx}}\right)_x = 0.$$

If there exist two linearly independent solutions $\tau_1(t)$ and $\tau_2(t)$, then

$$\left(\frac{f_{txx}}{f_{xxx}}\right)_x = 0 \qquad \text{and} \qquad \left(\frac{x f_{xxx} + 3 f_{xx}}{f_{xxx}}\right)_x = 0.$$

From these condition we obtain

$$f(t,x) = \frac{C}{x + H(t)} + F(t)x + G(t),$$

where $F(t)$, $G(t)$ and $H(t)$ are arbitrary functions and $C \neq 0$ is a constant.

We continue with an arbitrary constant C ($C = 0$ is included to incorporate the case $f_{xx} = 0$). Substitution into equation (4.4) gives

$$C(2A - \tau' H + 2\tau H') = 0$$

that leads to two cases.

(1) $C = 0$.
For $f(t,x) = F(t)x + G(t)$ the system of determining equations (4.2) and (4.3) has three linearly independent solutions $(\tau(t), A(t))$. We conclude that there exists a three-dimensional symmetry group admitted by the SDE.

(2) $C \neq 0$ and $A = \frac{\tau'}{2}H - \tau H'$.
Substitution into equations (4.2) and (4.3) gives the system

$$\frac{\tau''}{2} = \tau' F + \tau F', \qquad \frac{\tau'}{2}(H' - HF + G) + \tau(H' - HF + G)' = 0.$$

For the existence of two linearly independent solutions $\tau_1(t)$ and $\tau_2(t)$ we need the condition

$$G = -H' + HF$$

to hold.

We summarize the results as the following theorem.

Theorem 4.1. *The stochastic differential equation (4.1) admits a two-dimensional symmetry group if and only if*

$$f(t,x) = \frac{C}{x + H(t)} + F(t)(x + H(t)) - H'(t), \qquad C \neq 0 \tag{4.6}$$

and a three-dimensional symmetry group if and only if

$$f(t,x) = F(t)x + G(t). \tag{4.7}$$

Here $F(t)$, $G(t)$ and $H(t)$ are arbitrary functions and C is a constant.

The original SDE (2.1) can be transformed into an equation of the form (4.1) by the change of variables:

$$y = \int \frac{\mathrm{d}x}{g(t,x)}. \tag{4.8}$$

We obtain the equation

$$\mathrm{d}y = \left(-\int \frac{g_t}{g^2}\mathrm{d}x + \frac{f}{g} - \frac{g_x}{2} \right) \mathrm{d}t + \mathrm{d}W(t).$$

Applying condition (4.7) to the transformed equation, we get

$$-\int \frac{g_t}{g^2}\mathrm{d}x + \frac{f}{g} - \frac{g_x}{2} = F(t)\int \frac{\mathrm{d}x}{g(t,x)} + G(t).$$

Excluding the arbitrary functions by differentiation with respect to x, we conclude the following result.

Corollary 4.2. *The stochastic differential equation (2.1) admits a three-dimensional symmetry group if and only if its drift and diffusion coefficient satisfy equation (3.13).*

This condition was already obtained in theorem 3.4 by other reasoning.

Remark 4.3. Group classification of one-dimensional Fokker–Planck equation was given in [18]. For the drifts (4.6) and (4.7) the corresponding FP equations admit four- and six-dimensional symmetry groups, respectively. However, not all cases when FP equations admit four- and six-dimensional symmetry groups are presented by (4.6) and (4.7).

It is possible to provide a change of variables which transform stochastic differential equation (4.1) for drift coefficients (4.6) and (4.7) to the canonical forms of SDEs admitting two- and three-dimensional symmetry groups, presented in table 1. These transformations are

$$\bar{t}(t) = \int [\mathrm{e}^{-2\int F(t)\,\mathrm{d}t}]\,\mathrm{d}t, \qquad \bar{x}(t,x) = (x + H(t))\,\mathrm{e}^{-\int F(t)\,\mathrm{d}t} \tag{4.9}$$

and

$$\bar{t}(t) = \int [\mathrm{e}^{-2\int F(t)\,\mathrm{d}t}]\,\mathrm{d}t, \qquad \bar{x}(t,x) = x\,\mathrm{e}^{-\int F(t)\,\mathrm{d}t} - \int [\mathrm{e}^{-\int F(t)\,\mathrm{d}t}] G(t)\,\mathrm{d}t, \tag{4.10}$$

respectively.

In the rest of the section we will consider a number of examples.

Example 4.1. Interest rate models.

Let us consider the generalized one-factor interest rate model

$$\mathrm{d}x = (\alpha + \beta x)\,\mathrm{d}t + \sigma x^{\gamma}\,\mathrm{d}W(t), \qquad \sigma \neq 0, \tag{4.11}$$

where the parameters α, β, γ and σ are constants. It is reasonable to restrict ourselves to the case $x(t) > 0$. The description of particular models of this form can be found in [20, 21]. Such models are widely used in financial mathematics. An empirical comparison of eight short-term interest rate models of the form (4.11) was performed in [22].

The model equation (4.11) can be transformed into equation (4.1) by the change of variables

$$y = \int \frac{\mathrm{d}x}{\sigma x^{\gamma}} = \begin{cases} \dfrac{1}{\sigma}\dfrac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \\ \dfrac{1}{\sigma}\ln x, & \gamma = 1. \end{cases}$$

For these subcases the transformed equation takes the forms

$$\mathrm{d}y = \left(\alpha\sigma^{\frac{1}{\gamma-1}}((1-\gamma)y)^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)y + \frac{\gamma}{2(\gamma-1)}\frac{1}{y} \right)\mathrm{d}t + \mathrm{d}W(t), \qquad \gamma \neq 1$$

and

$$\mathrm{d}y = \left(\frac{\alpha}{\sigma}\mathrm{e}^{-\sigma y} + \left(\frac{\beta}{\sigma} - \frac{\sigma}{2} \right) \right)\mathrm{d}t + \mathrm{d}W(t), \qquad \gamma = 1.$$

Using theorem 4.1, we conclude the dimensionality of the admitted symmetry group without the actual computation of the symmetry operators. The results are the following.

(1) Three-dimensional symmetry group
  - (a) $\gamma = 0$,
  - (b) $\gamma = \frac{1}{2}$ and $\alpha = \frac{\sigma^2}{4}$,
  - (c) $\gamma = 1$ and $\alpha = 0$.

(2) Two-dimensional symmetry group
  - (a) $\gamma \neq \left\{0, \frac{1}{2}, 1\right\}$ and $\alpha = 0$,
  - (b) $\gamma = \frac{1}{2}$ and $\alpha \neq \frac{\sigma^2}{4}$.

(3) One-dimensional symmetry group
All other cases of the coefficients α, β and γ. (It is easy to see that for any values of the coefficients equation (4.11) admits time translations.)

Among the cases which are singled out one can see some popular models of interest rates. Vasicek model [23]: $\gamma = 0$; its particular case—Merton's model [24]: $\gamma = \beta = 0$; Cox–Ingersoll–Ross model [25]: $\gamma = \frac{1}{2}$; and Dothan's model [26]: $\gamma = 1$ and $\alpha = 0$.

Let us consider the equations admitting three-dimensional symmetry groups in greater detail. For $\gamma = 0$, equation (4.11) takes the form

$$\mathrm{d}x = (\alpha + \beta x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t). \tag{4.12}$$

It admits the following symmetry groups depending on the value of parameter β.

(1) $\beta \neq 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \mathrm{e}^{2\beta t}\frac{\partial}{\partial t} + (\alpha + \beta x)\,\mathrm{e}^{2\beta t}\frac{\partial}{\partial x}, \qquad X_3 = \mathrm{e}^{\beta t}\frac{\partial}{\partial x};$$

(2) $\beta = 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 2t\frac{\partial}{\partial t} + (x + \alpha t)\frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial x}.$$

By change of variables

$$y = \mathrm{e}^{-\beta t}x$$

equation (4.12) is transformed into the equation

$$\mathrm{d}y = \alpha\,\mathrm{e}^{-\beta t}\,\mathrm{d}t + \sigma\,\mathrm{e}^{-\beta t}\,\mathrm{d}W(t), \tag{4.13}$$

which can be easily integrated. For the original equation we get the solution

$$x(t) = \mathrm{e}^{\beta(t-t_0)}x(t_0) + \frac{\alpha}{\beta}(\mathrm{e}^{\beta(t-t_0)} - 1) + \sigma\int_{t_0}^{t}\mathrm{e}^{\beta(t-s)}\,\mathrm{d}W(s), \qquad \beta \neq 0$$

and

$$x(t) = x(t_0) + \alpha(t - t_0) + \sigma(W(t) - W(t_0)), \qquad \beta = 0.$$

For the case $\gamma = \frac{1}{2}$ and $\alpha = \frac{\sigma^2}{4}$ we get the equation

$$\mathrm{d}x = \left(\frac{\sigma^2}{4} + \beta x\right)\mathrm{d}t + \sigma\sqrt{x}\,\mathrm{d}W(t), \tag{4.14}$$

which admits the following symmetries.

(1) $\beta \neq 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \mathrm{e}^{\beta t}\frac{\partial}{\partial t} + \beta x\,\mathrm{e}^{\beta t}\frac{\partial}{\partial x}, \qquad X_3 = \sqrt{x}\,\mathrm{e}^{\beta t/2}\frac{\partial}{\partial x};$$

(2) $\beta = 0$

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_3 = \sqrt{x}\frac{\partial}{\partial x}.$$

The change of variables

$$y = \mathrm{e}^{-\beta t/2}\sqrt{x}$$

brings equation (4.14) into the form

$$\mathrm{d}y = \frac{\sigma}{2}\,\mathrm{e}^{-\beta t/2}\,\mathrm{d}W(t). \tag{4.15}$$

One can easily integrate it and find the solution of the original problem as

$$x(t) = \mathrm{e}^{\beta(t-t_0)}\left(\sqrt{x(t_0)} + \frac{\sigma}{2}\int_{t_0}^{t}\mathrm{e}^{\beta(t_0-s)/2}\,\mathrm{d}W(s)\right)^2.$$

In the last case with $\gamma = 1$ and $\alpha = 0$ we obtain the equation

$$\mathrm{d}x = \beta x \,\mathrm{d}t + \sigma x \,\mathrm{d}W(t), \tag{4.16}$$

which is well-known as the equation of geometric Brownian motion. It admits symmetries

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + \left(\ln x + \left(\beta - \frac{\sigma^2}{2}\right) t\right) x\frac{\partial}{\partial x}.$$

The change of variables

$$y = \ln x$$

transforms this equation into

$$\mathrm{d}y = \left(\beta - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma \,\mathrm{d}W(t). \tag{4.17}$$

By integration we find the solution of the original equation as

$$x(t) = x(t_0) \exp\left(\left(\beta - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma (W(t) - W(t_0))\right).$$

Of course, equations (4.13), (4.15) and (4.17) can be transformed to the equation of Brownian motion by the change of variables (2.14).

It is interesting to note that there are the same cases with two- and three-dimensional symmetry groups if we consider equation

$$\mathrm{d}x = (\alpha(t) + \beta(t)x)\,\mathrm{d}t + \sigma(t)x^\gamma \,\mathrm{d}W(t), \qquad \sigma(t) \not\equiv 0 \tag{4.18}$$

instead of (4.11). Here $\alpha \neq \frac{\sigma^2}{4}$ should be understood as $\alpha(t) = \rho\sigma^2(t)$, where $\rho \neq \frac{1}{4}$ is a constant. The cases with one-dimensional symmetry group cannot be characterized with the help of theorem 4.1. Generally, there is one admitted symmetry for other values of functions $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ which satisfy additional conditions. These conditions can be obtained from the determining equations. There are cases when equation (4.18) has no symmetries.

Example 4.2. A Brownian bridge [27] from α to β is a process governed by the equation

$$\mathrm{d}x = \frac{\beta - x}{T - t}\,\mathrm{d}t + \mathrm{d}W(t), \qquad 0 < t < T. \tag{4.19}$$

For any initial value $x(0) = \alpha$ the process has the final value $x(T) = \beta$. By change of variables (4.10), namely

$$\bar{t} = \frac{1}{T - t}, \qquad \bar{x} = \frac{x - \beta}{T - t},$$

equation (4.19) is transformed into the equation of Brownian motion. Taking the solution of the Brownian motion equation and applying the inverse change of variables, one gets the solution of (4.19) as

$$x(t) = \alpha\left(1 - \frac{t}{T}\right) + \beta\frac{t}{T} + (T - t)\int_0^t \frac{1}{T - s}\,\mathrm{d}W(s).$$

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